RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FOURTH SEMESTER EXAMINATION, MAY-JUNE 2013

SECOND YEAR

Date : 20/5/2013 MATHEMATICS (Honours)

Time : 11 am – 3 pm Paper : IV Full Marks : 100

[Use separate Answer Books for each group]

Group – A

Answer **any five** questions from **Q. No. 1-8** and **any three** questions from **Q. No. 9-13** :

1. a) Let (X,d) be a metric space such that $\text{Bd}A = \phi\ \forall A \subseteq X$. What can you say about $\tau(d)$, the topology induced by d?
   b) Let G be a noncyclic subgroup of ($\mathbb{R}$,+). Prove that G is dense in $\mathbb{R}$. [2+5]

2. a) Prove that is separable metric space is 2^{nd} countable.
   b) Assume that $\mathbb{R}$ with Euclidean metrice is Lindolöff. Prove that a subset of $\mathbb{R}$ which has no limit point is atmost countable. [4+3]

3. a) Let $\{x_n\}$ be a Cauchy sequence in a metric space (X,d). Prove that if $\{x_n\}$ has a convergent subsequence then $\{x_n\}$ is convergent.
   b) Prove that A is compact in (X,d) implies A is closed and bounded. Give an example to show that a closed and bounded subset in a metric space may not be compact. [3+4]

4. Let $A_1 \supseteq A_2 \supseteq ... A_n \supseteq A_{n+1} \supseteq ...$ be a decreasing sequence of closed subsets of a metric space (X,d).
   a) Is $\bigcap_{n=1}^{\infty} A_n \neq \phi$? Justify your answer.
   b) Show that if (X,d) $= (\mathbb{R}, d_u)$ where d_u is the Euclidean metric on $\mathbb{R}$ and $d(A_n) \to 0$ as $n \to \infty$ then $\bigcap_{n=1}^{\infty} A_n \neq \phi$.
   c) Prove that if (X,d) is compact then $\bigcap_{n=1}^{\infty} A_n \neq \phi$. [2+3+2]

5. a) State and prove Baire's Category theorem.
   b) Use Baire's Category theorem to prove that $\mathbb{R}^2$ can't be expressed as a countable union of straight lines in $\mathbb{R}^2$. [5+2]

6. a) If A, B are two disjoint closed subsets of a metric space (X,d), prove that there exists a continuous function $f : X \to R$ with f(A) = {0}, f(B) = {1} and $0 \le f(x) \le 1$ for all $x \in X$.
   b) Let $f, g : (X, d) \to (Y, d')$ be two continuous maps. Prove that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.
   Hence prove that if $f : (X, d) \to (X, d)$ is continuous then $\{x \in X : f(x) = x\}$ is closed in X. [3+4]

7. a) Prove that a totally bounded metric space is bounded.
   b) Show that a metric space (X,d) is totally bounded iff every sequence in X has a Cauchy subsequence. [2+5]

8. a) Prove that a continuous image of a connected metric space is connected.
   Use the above result to show that a connected metric space with atleast two distinct points is uncountable.
   b) Prove that a connected subgroup of ($\mathbb{R}$,+) is either {0} or $\mathbb{R}$. [4+3]

9. a) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued continuous functions on $E \subset \mathbb{R}$ converging uniformly to a function f over E. Prove that f is continuous on E.

b) Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ where $f_n(x)=\frac{x^n}{1+x^n}$, $x\in[0,2]$ is not uniformly convergent over [0,2] [3+2]

10. Let D be a compact subset of $\mathbb{R}$ and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions on D to $\mathbb{R}$, that converges pointwise to a continuous f on D. If the sequence $\{f_n\}_{n\in\mathbb{N}}$ is monotone of D, then prove that it converges uniformly on D to f. [5]

11. Let D be a subset of $\mathbb{R}$ and a series of functions $\sum_{n=1}^{\infty} f_n$ be uniformly convergent on D to a function f. Let x be a limit point of D and $\lim_{t\to x} f_n(t)=A_n, n=1,2,3,\ldots$ Prove that—

i) the series $\sum_{n=1}^{\infty} A_n$ is convergent and

ii) $\lim_{t\to x} f(t)$ exists and $\lim_{t\to x} f(t)=\sum_{n=1}^{\infty} A_n$. [5]

12. Show that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly over $\mathbb{R}$, here $f_n(x)=\frac{\sin nx}{n^2}$. Does $\sum_{n=1}^{\infty} f_n'(x)$ converge? Justify your answer. [3+2]

13. a) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{1+x^n} : x\ge 0$.

b) Find the radius of convergence of the power series $x+\frac{2^\alpha}{2!}x+\frac{3^\alpha}{3!}x^2+\ldots$ for $\alpha>0$. [3+2]

Group – B

Answer **any three** questions from **Q. No. 14-18** and **any four** questions from **Q. No. 19-24** :

14. a) Find the eigen-values and the corresponding eigenfunctions of $\frac{d^2y}{dx^2}+\lambda y=0\ (\lambda>0)$ under boundary conditions $y(0)+y'(0)=0$ and $y(1)+y'(1)=0$. [5]

b) Solve $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\sin 2x$ by reducing to normal form. [5]

15. a) Solve the following differential equation using Laplace transform
$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=4e^{2t}$, given $y(0)=-3$ and $y'(0)=5$. [5]

b) Knowing that y = x is a solution of the equation $x^2\frac{d^2y}{dx^2}-x(x+2)\frac{dy}{dx}+(x+2)y=0$ reduce the equation $x^2\frac{d^2y}{dx^2}-x(x+2)\frac{dy}{dx}+(x+2)y=x^3$ to a differential equation of 1st order and 1st degree and find its complete primitive. [5]

16. a) Find the integral surface of the linear partial differential equation $x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$ which contains the straight line x + y = 0, z = 1. [5]

b) Solve the equation $\frac{d^2y}{dx^2}+(x-1)^2\frac{dy}{dx}-4(x-1)y=0$ in series about the ordinary point x = 1. [5]

17. a) Find a complete integral of the partial differential equation $(p^2+q^2)y = qz$, using Charpit's method. [5]

b) Solve $(x^2-yz)p+(y^2-zx)q = z^2-xy$ by Lagrange's Method $\left(p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}\right)$. [5]

18. a) Use convolution theorem to show that $L^{-1}\left\{\frac{1}{(p+2)^2(p-2)}\right\} = \frac{1}{16}\left(e^{2t}-4te^{-2t}-e^{-2t}\right)$ where L^{-1} is inverse Laplace transformation. [3]

b) If F(t) be a periodic function with period T(>0) then show that $L\{F(t)\} = \frac{\int_0^T e^{-pt}F(t)dt}{1-e^{-pT}}$ where L represents Laplace transform operator. [4]

c) Form a partial differential equation by eliminating the arbitrary functions f and g from $z = yf(x)+xg(y)$ [3]

19. If $I_n = \int_0^{\pi/2} x^n \sin x\, dx$, n being a positive integer > 1, show that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$. Hence find the value of $\int_0^{\pi/2} x^5 \sin x dx$. [5]

20. Show that the pedal equation of $e^2(x^2+y^2) = x^2y^2$ w.r. to the origin is $\frac{1}{p^2}+\frac{3}{r^2}=\frac{1}{c^2}$. [5]

21. Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where the parameters a and b are connected by a relation $\frac{a^2}{\ell^2}+\frac{b^2}{m^2}=1$, ℓ and m are non-zero constants. [5]

22. Find the radius of curvature at the origin of the curve $4x^4+3y^3-8x^2y+2x^2-3xy-6y^2-8y=0$. [5]

23. Find the existence of double point, if any, of the curve $y^2(a^2+x^2) = x^2(a^2-x^2)$. If so, find its nature. [5]

24. State Pappus theorem on the volume of a solid of revolution and use it to find the volume of the solid generated by revolving the ellipse $x = a\cos\theta,\ y = b\sin\theta$ about the line x = 2a. [5]